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On the Approximation of Continuous Functions by Fourier–Legendre Sums

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Let $\{X_n\}_0^\infty$ be the orthonormal system of Legendre polynomials on $[-1, 1]$. For $f \in C[-1, 1]$ let $S_n(f)$ $(n+1 \in N)$ be the nth partial sum of the Fourier–Legendre series of the function f. Some refinements of the classical inequality

$$\|f - S_n(f)\|_{C[-1,1]} \leqslant A \cdot (n+1)^{1/2} E_n(f)_C, \qquad A = \text{const.}, \qquad A > 0, \tag{1}$$

involving best approximation in L_p-norms are discussed. For a class of examples we obtain better order estimates than those that can be derived from (1).

Furthermore, we show that the results are best possible in a certain sense. It turns out that only in two particular cases ($p = \frac{4}{3}$ and $p = 4$) there is no proof of optimality of the results.

In conclusion, we give without proof a generalization of the main theorem to the ultraspherical case. © 1996 Academic Press, Inc.

1. INTRODUCTION

Let L_p $(1 \leqslant p < \infty)$ be the space of functions f, measurable on $[-1, 1]$ and integrable to the power p, with norm

$$\|f\|_p = \left\{ \int_{-1}^{1} |f(x)|^p \, dx \right\}^{1/p}.$$

We set, by definition: (1) $L_\infty = C$, where C is the space of functions f, continuous on $[-1, 1]$, with $\|f\|_\infty = \|f\|_C = \max_{|x| \leqslant 1} |f(x)|$; (2) $L = L_1$. By N we mean the set of all natural numbers. Let $\{P_n\}_0^\infty$ be the system of Legendre polynomials normalized by the condition $P_n(1) = 1$, $n+1 \in N$.

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The Fourier–Legendre coefficients of a function $f \in C$ will be denoted by $c_k(f)$, i.e.,

$$c_k(f) = \int_{-1}^{1} f(x)\, X_k(x)\, dx, \qquad k+1 \in N.$$

We also set $R_n(f) = f - S_n(f)$ $(n+1 \in N)$. By H_n $(n+1 \in N)$ we denote the set of all algebraic polynomials of degree at most n, and for every $p \in [1, \infty]$, $n+1 \in N$, we set

$$E_n(f)_p = \inf\{\, \|f - Q_n\|_p : Q_n \in H_n\}.$$

By A and A with arguments between parentheses we denote (in general, different) absolute positive constants and positive constants depending on the arguments in the parentheses, respectively. For two sequences $\{\alpha_n\}_0^\infty$, $\{\beta_n\}_0^\infty$ of positive numbers, we shall write $\alpha_n \sim \beta_n$ if there exist constants $A_1, A_2 > 0$, independent of n such that

$$A_1 \beta_n \leqslant \alpha_n \leqslant A_2 \beta_n \qquad (n+1 \in N).$$

The idea of estimating in some metric the deviation of a linear polynomial operator from a given function, in terms of best approximation of the function by corresponding polynomials, goes back to Lebesgue [8]. The classical estimate (1) follows directly from results obtained in [5] and [19]. In [14], one of the authors established the following statement.

Let $\nu \geqslant 0$, $n+1 \in N$,

$$\|f\|_{\infty,\nu} = \max_{|x| \leqslant 1} |(1-x^2)^\nu f(x)|, \qquad E_n(f)_{\infty,\nu} = \inf\{\, \|f - Q_n\|_{\infty,\nu} : Q_n \in H_n\}.$$

For $f \in C$, $0 \leqslant \nu < \frac{1}{4}$, $n+1 \in N$ one has

$$\|R_n(f)\|_C \leqslant A(\nu)\{(n+1)^{2\nu} E_n(f)_C + (n+1)^{1/2} E_n(f)_{\infty,\nu}\} \tag{2}$$

(see also [21]).

The following estimate was proved in [17]: If $f \in C$, $p > 4$, $n+1 \in N$, then

$$\|R_n(f)\|_C \leqslant A(p)\{(n+1)^{2/p} E_n(f)_C + (n+1)^{1/2} E_n(f)_p\}. \tag{3}$$

Both inequalities (2) and (3) give no worse, and sometimes better, order estimates for $\|R_n(f)\|_C$ than the estimate (1).

In this paper we obtain further refinements of (1), similar to (3). As a result we derive estimates for $\|R_n(f)\|_C$ that for some particular functions are better with respect to order than those given by (1) and (3).

Some statements, similar to (3), for trigonometric Fourier series can be found in [25–27].

2. THE MAIN THEOREM

THEOREM 1. *Let $f \in C$, $n+1 \in N$. Then*

$$\|R_n(f)\|_C \leqslant \begin{cases} A(p)(E_n(f)_C + (n+1)^{2/3p} E_n^{1/3}(f)_p E_n^{2/3}(f)_C), & 1 \leqslant p < 4/3, \quad (4) \\ A(E_n(f)_C + (n+1)^{1/2} \ln^{1/12}(n+2) E_n^{1/3}(f)_p E_n^{2/3}(f)_C), & p = \frac{4}{3}, \quad (5) \\ A(p)(E_n(f)_C + (n+1)^{1/2} (E_n(f)_p)^{p/4} (E_n(f)_C)^{1-p/4}), & \frac{4}{3} < p < 4, \quad (6) \\ A(E_n(f)_C + (n+1)^{1/2} \ln^{3/4}(n+2) E_n(f)_p), & p = 4, \quad (7) \\ A(p)(E_n(f)_C + (n+1)^{1/2} E_n(f)_p), & p > 4. \quad (8) \end{cases}$$

Before proving this theorem, we discuss its relationship to the estimates (1) and (3), as well as possible applications. The estimate (8) is a further sharpening of (3); the estimate (6) is a refinement of (1). When applying Theorem 1 one can take into account that for some particular functions f the order of decrease of $E_n(f)_C$ and $E_n(f)_p$ is either known or can be estimated from above. It is worth noting that for this purpose one can make use of different Jackson-type theorems [9–11, 13, 15]. For example, as a consequence of (1) and of Jackson's theorem on the approximation of a function by algebraic polynomials in the C-metric, we obtain that for $f \in \operatorname{Lip} \gamma$, $\gamma > \frac{1}{2}$, one has the estimate

$$\|R_n(f)\|_C \leqslant A(\gamma)(n+1)^{1/2-\gamma}, \qquad n+1 \in N.$$

We give two examples in order to illustrate possible applications of Theorem 1.

EXAMPLE 1. We consider the function $f(x) = (x-c)^{r-1} |x-c|$, $r \in N$, $|c| < 1$. It is known [23, p. 426] that

$$E_n(f)_q \leqslant A(r, q, c)(n+1)^{-r-1/q}, \qquad 1 \leqslant q \leqslant \infty, \quad n+1 \in N.$$

After applying (6), we obtain the estimate

$$\|R_n(f)\|_C \leqslant A(r, q, c)(n+1)^{-r+1/4}, \qquad n+1 \in N.$$

This estimate cannot be derived by making use of (1) or (3).

EXAMPLE 2. For the function $\varphi(x) = (1-x)^r \ln(1-x)$, $r \in N$, we have the relations

$$E_n(\varphi)_C \sim (n+1)^{-2r}, \qquad E_n(\varphi)_1 \leqslant A(r)(n+1)^{-2r-2}$$

(see [16, p. 462] and [18], respectively). After using (4) with $p=1$, we obtain

$$\|R_n(\varphi)\|_C \leqslant A(r)(n+1)^{-2r},$$

showing that in the C-metric the Fourier–Legendre partial sums provide for the function φ an approximation of best possible order.

3. THE MAIN TOOLS

We shall need the following facts from the theory of Jacobi polynomials. We have [3]:

$$\|X_n\|_p \sim \begin{cases} (n+1)^{1/2-2/p}, & p>4 \\ \ln^{1/4}(n+2), & p=4 \\ 1, & 1\leqslant p<4. \end{cases} \tag{9}$$

The following inequality holds [22, p. 168]:

$$(1-x^2)^{1/4}\,|X_n(x)| \leqslant A, \qquad (n+1\in N,\ |x|\leqslant 1). \tag{10}$$

Let $\{P_n^{(1,0)}\}_0^\infty$ be the system of Jacobi polynomials, orthogonal on $[-1,1]$ with respect to the weight function $1-x$ and normalized by the condition $P_n^{(1,0)}(1)=n+1$ $(n+1\in N)$. Then we have [22, pp. 168 and 71 respectively]

$$(1-x)^{3/4}\,(1+x)^{1/4}\,|P_n^{(1,0)}(x)| \leqslant A(n+1)^{-1/2}, \qquad n+1\in N, \quad |x|\leqslant 1; \tag{11}$$

$$\sum_{\nu=0}^{n} (2\nu+1)\,P_\nu = (n+1)\,P_n^{(1,0)} \tag{12}$$

(see also [22, p. 71]).

4. PROOF OF THEOREM 1

We can assume $f\notin H_n$ since otherwise our statement is obvious. For $n+1,\ k\in N$ we set

$$\sigma_{n,k}(f)=k^{-1}\sum_{l=n}^{n+k-1} S_l(f);$$

$\sigma_{n,k}(f)$ are the de La Vallée Poussin sums for f in the system $\{X_n\}_0^\infty$. Obviously,

$$\|R_n(f)\|_C \leqslant \|f-\sigma_{n,k}(f)\|_C + \|\sigma_{n,k}(f)-S_n(f)\|_C. \tag{13}$$

It was proved in [7] that for $n+1,\ k \in N$ one has

$$\|f-\sigma_{n,k}(f)\|_C \leqslant A\left(\frac{\sqrt{n+1}}{\sqrt{k}}+1\right) E_n(f)_C. \tag{14}$$

In order to estimate the second term of the right-hand side of (13), we make use of the easily verifiable equality

$$\sigma_{n,k}(f)-S_n(f)=k^{-1}\sum_{j=n+1}^{n+k-1}(n+k-j)\,c_j(f)\,X_j \tag{15}$$

(for $k=1$ we assume that the last sum is equal to 0).

We consider now the generalized translation operator (g.t.o.) $f \to f_t$ defined on L by

$$f_t(x)=\pi^{-1}\int_0^\pi f(x\cos t+\sqrt{1-x^2}\sin t\cos\lambda)\,d\lambda, \qquad |t|\leqslant\pi. \tag{16}$$

This operator was introduced in [4] (see also [2, 24]). Obviously, $f \in C$ implies $f_t \in C$ ($|t| \leqslant \pi$). We shall use the following properties of the g.t.o.:

(1) for $f \in L,\ k+1 \in N,\ t \in [-\pi, \pi]$ we have [4, 2, 24]

$$c_k(f_t)=c_k(f)\,P_k(\cos t); \tag{17}$$

(2) for $f \in C,\ x \in [-1, 1],\ t \in [-\pi, \pi]$ we have

$$f_t(x)=f_{\arccos x}(\cos t); \tag{18}$$

(3) for $f \in L_p$ $(1 \leqslant p \leqslant \infty)$, $t \in [-\pi, \pi]$ we have

$$\|f_t\|_p \leqslant \|f\|_p. \tag{19}$$

Equality (18) is obvious. Inequality (19) was proved for $p=1$ in [16]; for $1<p<\infty$ it can be proved in a similar manner; for $p=\infty$ it is obvious. For $n+1,\ k \in N$ we introduce the function

$$\Phi_{n,k}=\begin{cases} k^{-1}\sum_{j=n+1}^{n+k-1}(n+k-j)\,X_j(1)\,X_j, & k \geqslant 2,\\ 0, & k=1. \end{cases} \tag{20}$$

We show now that for $n+1$, $k\in N$, $x\in[-1,1]$ we have

$$\sigma_{n,k}(f;x)-S_n(f;x)=\int_0^\pi f_t(x)\,\Phi_{n,k}(\cos t)\sin t\,dt. \tag{21}$$

For $k=1$ the equality (21) is obvious. Assume that $k\geqslant 2$. Taking into account (18), (17), (20), and (15), we obtain

$$\begin{aligned}
&\int_0^\pi f_t(x)\,\Phi_{n,k}(\cos t)\sin t\,dt\\
&\quad=\int_0^\pi f_{\arccos x}(\cos t)\,\Phi_{n,k}(\cos t)\sin t\,dt\\
&\quad=\int_{-1}^1 \mathrm{f}_{\arccos x}(z)\,\Phi_{n,k}(z)\,dz=\sum_{j=0}^{\infty} c_j(f_{\arccos x})\,c_j(\Phi_{n,k})\\
&\quad=k^{-1}\sum_{j=n+1}^{n+k-1} c_j(f)(n+k-j)\,X_j=\sigma_{n,k}(f)-S_n(f).
\end{aligned}$$

Suppose that $Q_n\in H_n$, $\|f-Q_n\|_p=E_n(f)_p$. It follows from (17) that $Q_{n,t}\in H_n$ $\forall t\in[-\pi,\pi]$ and, therefore, according to (20) and (21) we obtain

$$\sigma_{n,k}(f;x)-S_n(f;x)=\int_0^\pi [f_t(x)-Q_{n,t}(x)]\,\Phi_{n,k}(\cos t)\sin t\,dt. \tag{22}$$

Applying Hölder's inequality, in view of (19) we get

$$\|\sigma_{n,k}(f)-S_n(f)\|_C\leqslant\|\Phi_{n,k}\|_q\cdot E_n(f)_p,\qquad p^{-1}+q^{-1}=1. \tag{23}$$

For the estimation of $\|\Phi_{n,k}\|_q$ we shall consider in turn all the five possibilities for p, listed in the formulation of Theorem 1.

(i) $1\leqslant p<\frac{4}{3}$. Making use of the first inequality in (9), we have

$$\begin{aligned}
\|\Phi_{n,k}\|_q&\leqslant k^{-1}\sum_{j=n+1}^{n+k-1}(n+k-j)\,X_j(1)\,\|X_j\|_q\\
&\leqslant A(p)\,k^{-1}\sum_{j=n+1}^{n+k-1}(n+k-j)\,j^{1-2/q}\\
&\leqslant A(p)\,k^{-1}(n+k-1)^{1-2/q}\,k(k-1)\\
&=A(p)(n+k-1)^{2/p-1}\,(k-1).
\end{aligned}$$

It follows from (13), (14), (23), and (24) that for $k \leqslant n+2$ we have

$$\|R_n(f)\|_C \leqslant A\left(\frac{\sqrt{n+1}}{\sqrt{k}}+1\right) E_n(f)_C$$
$$+A(p)((n+1)^{2/p-1}(k-1)\,E_n(f)_p+(k-1)^{2/p}\,E_n(f)_p)$$
$$\leqslant A(p)(\varphi_1(k)+\psi_1(k))=A(p)\,\lambda_1(k), \tag{25}$$

where

$$\varphi_1(k)=\left(\frac{\sqrt{n+1}}{\sqrt{k}}+1\right) E_n(f)_C,$$
$$\psi_1(k)=\begin{cases}(n+1)^{2/p-1}(k-1)\,E_n(f)_p, & k\leqslant n+2\\ (k-1)^{2/p}\,E_n(f)_p, & k>n+2,\end{cases} \qquad \lambda_1(k)=\varphi_1(k)+\psi_1(k). \tag{26}$$

Further

$$\|R_n(f)\|_C \leqslant A(p)\cdot \min_{k\leqslant n+2} \lambda_1(k). \tag{27}$$

The evaluation of $\min_{k\leqslant n+2}\lambda_1(k)$ is a simple exercise in calculus. From $d\lambda_1(k)/dk=0$ we obtain

$$k=k_0=(n+1)^{1-4/3p}\,(E_n(f)_C)^{2/3}\,(E_n(f)_p)^{-2/3}.$$

Now we consider two cases:

(a) $k_0\leqslant n+2$; in this case

$$\min_{k\leqslant n+2}\lambda_1(k)\leqslant\lambda_1([k_0]+1)$$
$$\leqslant\left(\frac{\sqrt{n+1}}{\sqrt{k_0}}+1\right)E_n(f)_C+(n+1)^{2/p-1}\,k_0E_n(f)_p$$
$$=E_n(f)_C+2(n+1)^{2/3p}\,E_n^{1/3}(f)_p\,E_n^{2/3}(f)_C; \tag{28}$$

here and in the following by $[a]$ we mean the greatest integer not exceeding a.

(b) $k_0>n+2$; in this case

$$\min_{k\leqslant n+2}\lambda_1(k)=\lambda_1(n+2)\leqslant 2E_n(f)_C+(n+1)^{2/p}\,E_n(f)_p.$$

We note that the inequality $k_0 > n+2$ implies $(n+1)^{2/p} E_n(f)_p < E_n(f)_C$, so that

$$\min_{k \leqslant n+2} \lambda_1(k) \leqslant 3E_n(f)_C. \tag{29}$$

Inequality (4) follows directly from (27)–(29).

(ii) $p = \frac{4}{3}$. It is easy to prove (5) by following the same line of reasoning as the one considered in case (i). The only difference is that instead of the first estimate in (9) we make use of the second one.

(iii) $\frac{4}{3} < p < 4$. We estimate $\|\Phi_{n,k}\|_q$ in the following manner:

$$\|\Phi_{n,k}\|_q \leqslant \left\{\int_0^{k^{-1}} |\Phi_{n,k}(\cos t)|^q \sin t\, dt\right\}^{1/q} + \left\{\int_{k^{-1}}^{\pi} |\Phi_{n,k}(\cos t)|^q \sin t\, dt\right\}^{1/q}$$

$$= I_1 + I_2. \tag{30}$$

Making use of (10) and observing that $q < 4$ we obtain

$$\begin{aligned} I_1 &\leqslant Ak^{-1} \sum_{j=n+1}^{n+k-1} (n+k-j)\, j^{1/2} \left\{\int_0^{k^{-1}} (\sin t)^{1-q/2}\, dt\right\}^{1/q} \\ &\leqslant A(p)\, k^{-1} k^{1/2-2/q} (n+k-1)^{1/2}\, k(k-1) \\ &= A(p)(k-1)\, k^{2/p-3/2} (n+k-1)^{1/2}. \end{aligned} \tag{31}$$

By applying Abel's transformation, in view of (12) we obtain

$$\sum_{j=n+1}^{n+k-1} (n+k-j)\, X_j(1)\, X_j(\cos t)$$

$$= -\tfrac{1}{2}(k-1)(n+1)\, P_n^{(1,0)}(\cos t) + \tfrac{1}{2} \sum_{j=n+1}^{n+k-1} (j+1)\, P_j^{(1,0)}(\cos t).$$

Further,

$$\begin{aligned} I_2 &\leqslant (2k)^{-1} \left\{\int_{k^{-1}}^{\pi} \left|\sum_{j=n+1}^{n+k-1} (j+1)\, P_j^{(1,0)}(\cos t)\right|^q \sin t\, dt\right\}^{1/q} \\ &\quad + (2k)^{-1} (k-1)(n+1) \left\{\int_{k^{-1}}^{\pi} |P_n^{(1,0)}(\cos t)|^q \sin t\, dt\right\}^{1/q} \\ &= (2k)^{-1} I_{21} + (k-1)(n+1)(2k)^{-1} I_{22}. \end{aligned} \tag{32}$$

Taking into account (11), we derive

$$\begin{aligned} I_{22} &\leqslant \left\{\int_{k^{-1}}^{\pi/2} |P_n^{(1,0)}(\cos t)|^q \sin t\, dt\right\}^{1/q} + \left\{\int_{\pi/2}^{\pi} |P_n^{(1,0)}(\cos t)|^q \sin t\, dt\right\}^{1/q} \\ &\leqslant A(p)(n+1)^{-1/2} \left\{\int_{k^{-1}}^{\pi/2} t^{-3q/2+1}\, dt\right\}^{1/q} \\ &\quad + A(p)(n+1)^{-1/2} \left\{\int_{\pi/2}^{\pi} (\pi - t)^{-q/2+1}\, dt\right\}^{1/q} \\ &\leqslant A(p)(n+1)^{-1/2}\, k^{3/2-2/q}. \end{aligned} \tag{33}$$

In order to estimate I_{21} we make use of (33):

$$\begin{aligned} I_{21} &\leqslant \sum_{j=n+1}^{n+k-1} (j+1) \left\{\int_{k^{-1}}^{\pi} |P_j^{(1,0)}(\cos t)|^q \sin t\, dt\right\}^{1/q} \\ &\leqslant A(p)\, k^{3/2-2/q} \sum_{j=n+1}^{n+k-1} (j+1)^{1/2} \leqslant A(p)\, k^{3/2-2/q}(n+k)^{1/2}\, (k-1). \end{aligned} \tag{34}$$

From (30)–(34) it follows that

$$\|\Phi_{n,k}\|_q \leqslant A(p)(k-1)(n+k)^{1/2}\, k^{1/2-2/q}. \tag{35}$$

Taking into account (13), (14), (23), and (35), we obtain

$$\begin{aligned} \|R_n(f)\|_C &\leqslant A\left(1 + \frac{\sqrt{n+1}}{\sqrt{k}}\right) E_n(f)_C + A(p)(k-1)\, k^{1/2-2/q} n^{1/2} E_n(f)_p \\ &\quad + A(p)(k-1)\, k^{1-2/q} E_n(f)_p. \end{aligned} \tag{36}$$

If in (36) we set $k = [E_n^{p/2}(f)_C \cdot E_n^{-p/2}(f)_p] + 1$, then, after some simplification, we obtain (6).

(iv) $p = 4$. With the same reasoning as that used in case (iii), we obtain

$$\begin{aligned} \|R_n(f)\|_C \leqslant A\{&(1 + (n+1)^{1/2}\, k^{-1/2})\, E_n(f)_C \\ &+ k^{-1}(k-1) \ln^{3/4}(k+1)(n+1)^{1/2}\, E_n(f)_4 \\ &+ k^{-1/2}(k-1) \ln^{3/4}(k+1) \cdot E_n(f)_4\}. \end{aligned} \tag{37}$$

To obtain (7) it remains to set $k = n+1$ in (37).

(v) $p>4$. In the same way as that in case (iii) we derive that

$$\begin{aligned}\|R_n(f)\|_C \leqslant A(p)\{&(1+(n+1)^{1/2}k^{-1/2})\,E_n(f)_C\\ &+((k-1)\,k^{2/p-3/2}(n+k-1)^{1/2}\\ &+k^{-1}(k-1)(n+k)^{1/2})\,E_n(f)_p\}.\end{aligned} \tag{38}$$

Setting $k=n+1$ in (38), we obtain (8). This completes the proof of Theorem 1.

5. SOME AUXILIARY STATEMENTS

In order to analyze the sharpness of the estimates obtained in Theorem 1 we need the following lemmas.

LEMMA 1 [1, p. 325]. *If $a>1$, $n+1\in N$, then*

$$E_n((a-x)^{-1/2})_C=\frac{\Gamma(n+3/2)}{\pi^{1/2}\Gamma(n+2)}\cdot\frac{(a-\sqrt{a^2-1})^n}{(a^2-1)^{3/4}}(1+\delta_n(a)), \tag{39}$$

where Γ is Euler's gamma function and

$$|\delta_n(a)|\leqslant\frac{5}{4(n+2)\sqrt{a^2-1}}, \tag{40}$$

if the right-hand side of (40) *is less than* 1.

LEMMA 2 [20]. *There exists $f^*\in C$ such that for infinitely many values of $n\in N$ we have*

$$\|R_n(f^*)\|_C\geqslant A(n+1)^{1/2}\,E_n(f^*)_C. \tag{41}$$

LEMMA 3 [6]. *If $f\in L_p$, $1\leqslant p<q\leqslant\infty$, and*

$$\sum_{n=1}^{\infty} n^{2(1/p-1/q)-1}E_n(f)_p<\infty,$$

then $f\in L_q$ and

$$E_n(f)_q\leqslant A(p,q)\left\{n^{2(1/p-1/q)}E_n(f)_p+\sum_{\nu=n+1}^{\infty}\nu^{2(1/p-1/q)-1}E_\nu(f)_p\right\}. \tag{42}$$

LEMMA 4. *For $1<a\leqslant 2$ and*

$$m=n+2\geqslant[(a-1)^{-1/2}]+1 \tag{43}$$

we have

$$E_n((a-x)^{-1/2})_1 \leqslant A \cdot \frac{(a-1)^{1/4}}{\sqrt{m}} \cdot (a-\sqrt{a^2-1})^m. \tag{44}$$

Proof. The following equality is known [1, p. 327]:

$$E_n((a-x)^{-1/2})_1 = \frac{2}{\pi}\int_a^{\infty} (t-a)^{-1/2} \ln \frac{(t+\sqrt{t^2-1})^{n+2}+1}{(t+\sqrt{t^2-1})^{n+2}-1}\, dt. \tag{45}$$

Making the change of variable $u = t-(t^2-1)^{1/2}$, we obtain

$$E_n((a-x)^{-1/2})_1 = 2^{1/2}\pi^{-1}\int_0^{a-\sqrt{a^2-1}} (1-u^2)\, u^{-3/2}(a+\sqrt{a^2-1}-u)^{-1/2}$$
$$\times (a-\sqrt{a^2-1}-u)^{-1/2} \ln \frac{1+u^m}{1-u^m}\, du.$$

Since for $|u|<1$ we have

$$\ln \frac{1+u^m}{1-u^m} = 2u^m \sum_{k=0}^{\infty} \frac{u^{2km}}{2k+1},$$

it follows that

$$E_n((a-x)^{-1/2})_1 \leqslant 2^{5/2}\pi^{-1} \sum_{k=0}^{\infty} (2k+1)^{-1} \int_0^{a-\sqrt{a^2-1}} (1-u)\, u^{(2k+1)m-3/2}$$
$$\times ((a+\sqrt{a^2-1}-u)(a-\sqrt{a^2-1}-u))^{-1/2}\, du$$
$$= \frac{2^{5/2}}{\pi} \sum_{k=0}^{\infty} \frac{I_{1;k,m,a}-I_{0;k,m,a}}{2k+1}, \tag{46}$$

where

$$I_{j;k,m,a} = \int_0^{a-\sqrt{a^2-1}} \frac{u^{(2k+1)m-1/2-j}\, du}{(a+\sqrt{a^2-1}-u)^{1/2}\,(a-\sqrt{a^2-1}-u)^{1/2}}, \qquad (j=0,1). \tag{47}$$

In order to express $I_{j;k,m,a}$ as an infinite series, we make use of the following formula [12, p. 301]:

$$\int_a^b (x-a)^{\alpha-1}\,(b-x)^{\beta-1}\,(cx+d)^{\gamma}\, dx$$
$$= (b-a)^{\alpha+\beta-1}(ac+d)^{\gamma}\, B(\alpha,\beta)\cdot {}_2F_1\left(\alpha,\, -\gamma;\, \alpha+\beta;\, \frac{c(a-b)}{ac+d}\right), \tag{48}$$

where

$$\alpha, \beta > 0, \qquad \left| \arg \frac{d+cb}{d+ca} \right| < \pi, \qquad B(\alpha, \beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

$$_2F_1(a_1, a_2; b_1; z) = \sum_{i=0}^{\infty} \frac{(a_1)_i\,(a_2)_i}{(b_1)_i} \cdot \frac{z^i}{i!}, \qquad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)}.$$

$_2F_1(\,\bullet\,,\ \bullet\,;\bullet;\bullet)$ is the hypergeometric function. Applying formula (48) to both integrals (47), we obtain

$$I_{1;k,m,a} = (a-\sqrt{a^2-1})^{(2k+1)\,m-1/2} \sum_{i=0}^{\infty} \frac{\Gamma(i+1/2)}{\Gamma(i)} \times \frac{\Gamma((2k+1)\,m-1/2+i)}{\Gamma((2k+1)\,m+i)} (a-\sqrt{a^2-1})^{2i}, \tag{49}$$

$$I_{0;k,m,a} = (a-\sqrt{a^2-1})^{(2k+1)\,m+1/2} \sum_{i=0}^{\infty} \frac{\Gamma(i+1/2)}{\Gamma(i+1)} \times \frac{\Gamma((2k+1)\,m+1/2+i)}{\Gamma((2k+1)\,m+i+1)} (a-\sqrt{a^2-1})^{2i}. \tag{50}$$

Now it is easy to see that

$$\begin{aligned} &I_{1;k,m,a} - I_{0;k,m,a} \\ &\quad = (a-\sqrt{a^2-1})^{(2k+1)\,m-1/2} \sum_{i=0}^{\infty} \frac{\Gamma(i+1/2)}{\Gamma(i+1)} \frac{\Gamma((2k+1)\,m-1/2+i)}{\Gamma((2k+1)\,m+i)} \\ &\qquad \times (a-\sqrt{a^2-1})^{2i} \left[1-(a-\sqrt{a^2-1}) \cdot \frac{(2k+1)\,m-1/2+i}{(2k+1)\,m+i} \right] \\ &\quad \leqslant A(a-\sqrt{a^2-1})^{(2k+1)\,m-1/2} \,((2k+1)\,m)^{-1/2}\,(a-1)^{1/2} \\ &\qquad \times \sum_{i=0}^{\infty} \frac{(a-\sqrt{a^2-1})^{2i}}{(i+1)^{1/2}} + A(a-\sqrt{a^2-1})^{(2k+1)\,m+1/2}\,((2k+1)\,m)^{-1/2} \\ &\qquad \times \sum_{i=0}^{\infty} \frac{(a-\sqrt{a^2-1})^{2i}}{(i+1)^{1/2}} \left[1-\frac{(2k+1)\,m-1/2+i}{(2k+1)\,m+i} \right] \\ &\quad \leqslant A(a-\sqrt{a^2-1})^{(2k+1)\,m-1/2}\,((2k+1)m)^{-1/2}\,(a-1)^{1/2} \\ &\qquad \times \sum_{i=0}^{\infty} \frac{(a-\sqrt{a^2-1})^{2i}}{(i+1)^{1/2}} + A(a-\sqrt{a^2-1})^{(2k+1)\,m+1/2}\,(2k+1)^{-3/2}\,m^{-3/2} \\ &\qquad \times \sum_{i=0}^{\infty} \frac{(a-\sqrt{a^2-1})^{2i}}{(i+1)^{1/2}}. \end{aligned} \tag{51}$$

From (46) and (51) we derive

$$\begin{aligned} E_n((a-x)^{-1/2})_1 & \\ &\leqslant A(a-1)^{1/2} m^{-1/2} \cdot \sum_{k=0}^{\infty} (2k+1)^{-3/2} (a-\sqrt{a^2-1})^{(2k+1)m-1/2} \\ &\quad \times \sum_{i=0}^{\infty} \frac{(a-\sqrt{a^2-1})^{2i}}{(i+1)^{1/2}} + A \cdot m^{-3/2} \\ &\quad \times \sum_{k=0}^{\infty} (2k+1)^{-5/2} (a-\sqrt{a^2-1})^{(2k+1)m+1/2} \sum_{i=0}^{\infty} \frac{(a-\sqrt{a^2-1})^{2i}}{(i+1)^{1/2}}. \end{aligned} \tag{52}$$

In order to estimate $S(a)=\sum_{i=0}^{\infty} (a-\sqrt{a^2-1})^{2i} (i+1)^{-1/2}$ from above we set $M=[(a-1)^{-1/2}]$ and split $S(a)$ into two sums:

$$S(a)=\sum_{i=0}^{M} + \sum_{i=M+1}^{\infty} = S_{1,M}(a)+S_{2,M}(a). \tag{53}$$

Obviously,

$$S_{1,M}(a) \leqslant \sum_{i=0}^{M} (i+1)^{-1/2} < 2(M+1)^{1/2} \leqslant 2\sqrt{2}(a-1)^{-1/4}. \tag{54}$$

Further,

$$\begin{aligned} S_{2,M}(a) &\leqslant (M+2)^{-1/2} \sum_{i=M+1}^{\infty} (a-\sqrt{a^2-1})^{2i} \\ &= (M+2)^{-1/2} (a-\sqrt{a^2-1})^{2(M+1)} (1-(a-\sqrt{a^2-1})^2)^{-1} \\ &\leqslant 2^{-1}(M+2)^{-1/2} (a^2-1)^{-1/2} (a-\sqrt{a^2-1})^{-1} \\ &\leqslant \frac{a+\sqrt{a^2-1}}{2(a-1)^{1/2} (M+2)^{1/2}} \leqslant 2(a-1)^{-1/4}. \end{aligned} \tag{55}$$

Combining the relations (53)–(55), we obtain

$$S(a) \leqslant A(a-1)^{-1/4}. \tag{56}$$

From (52) and (56), taking into account that $m \geqslant (a-1)^{-1/2}$, we obtain the inequality (44).

Remark. It can be proved that there exists $\varepsilon>0$ such that for $1<a\leqslant 1+\varepsilon$ and $m=n+2 \geqslant [(a-1)^{-1/2}]+1$ we have

$$E_n((a-x)^{-1/2})_1 \geqslant A(a-1)^{1/4} m^{-1/2}(a-\sqrt{a^2-1})^m.$$

6. ON THE SHARPNESS OF THE ESTIMATES (4)–(8)

THEOREM 2. *There exists no function $\omega(n)$ such that*

(a) $\omega(n) > 0$ $(n+1 \in N)$,

(b) $\omega(n) \to 0$ *as* $n \to \infty$, *and*

(c) *for all* $f \in C$, $n+1 \in N$

$$\|R_n(f)\|_C \leqslant A(p)(E_n(f)_C + \omega(n)(n+1)^{2/3p} E_n^{1/3}(f)_p\, E_n^{2/3}(f)_C), \qquad 1 \leqslant p < \tfrac{4}{3}, \tag{57}$$

or

$$\|R_n(f)\|_C \leqslant A(p)(E_n(f)_C + \omega(n)(n+1)^{1/2} E_n^{1-p/4}(f)_C\, E_n^{p/4}(f)_p), \quad \tfrac{4}{3} < p < 4, \tag{58}$$

or

$$\|R_n(f)\|_C \leqslant A(p)(E_n(f)_C + \omega(n)(n+1)^{1/2} E_n(f)_p), \qquad p > 4. \tag{59}$$

Remark. As a consequence of this theorem, we conclude, in particular, that for $1 \leqslant p < \frac{4}{3}$ and for $\frac{4}{3} < p < 4$ the estimate (8) does not hold for all $f \in C$ and $n+1 \in N$.

THEOREM 3. *There exists no* $\varepsilon > 0$ *such that for all* $f \in C$, $n+1 \in N$ *one has*

$$\|R_n(f)\|_C \leqslant A(p)(E_n(f)_C + (n+1)^{2/3p} E_n^{1/3+\varepsilon}(f)_p \cdot E_n^{2/3-\varepsilon}(f)_C), \quad 1 \leqslant p < \tfrac{4}{3}, \tag{60}$$

or

$$\|R_n(f)\|_C \leqslant A(p)(E_n(f)_C + (n+1)^{1/2} E_n^{1-p/4-\varepsilon}(f)_C \cdot E_n^{p/4+\varepsilon}(f)_p), \quad \tfrac{4}{3} < p < 4. \tag{61}$$

Proof of Theorem 2. Let $1 \leqslant p < \frac{4}{3}$. We assume, contrary to the assertion of the theorem, that there exists a function $\omega(n)$ satisfying the conditions (a), (b), and (c). We make use of the function $f_a(x) = (a-x)^{-1/2}$, $1 < a \leqslant 2$. If (43) holds, then

$$\frac{5}{4}(n+2)^{-1}(a^2-1)^{-1/2} < \frac{5}{4\sqrt{2}}(n+2)^{-1}(a-1)^{-1/2} \leqslant \frac{5}{4\sqrt{2}} < 1,$$

and therefore, due to Lemma 1, we have

$$E_n(f_a)_C \leqslant A(n+1)^{-1/2}(a-1)^{-3/4}(a-\sqrt{a^2-1})^n. \tag{62}$$

We set $a=(2z)^{-1}(z^2+1)$, $0<z<1$, $z=a-\sqrt{a^2-1}$. Then

$$f_a(x)=(2z)^{1/2}(z^2-2zx+1)^{-1/2}=(2z)^{1/2}\sum_{k=0}^{\infty}P_k(x)\,z^k. \tag{63}$$

Obviously,

$$\|R_n(f_a)\|_C=(2z)^{1/2}(1-z)^{-1}z^{n+1}. \tag{64}$$

From (42) and (44) it follows that, under the conditions of Lemma 4, we have

$$E_n(f_a)_p\leqslant A(p)(a-1)^{1/4}(n+1)^{3/2-2/p}(a-\sqrt{a^2-1})^m. \tag{65}$$

If in (57) we replace f by f_a and take into account (44) and (62)–(64) then, after some simplification, we obtain

$$1\leqslant A(p)\{z^{-3/4}(n+1)^{-1/2}(1-z)^{-1/2}+\omega(n)(n+1)^{1/6}(1-z)^{1/6}z^{-5/12}\}. \tag{66}$$

We set

$$\gamma(n)=\min\left\{\frac{n}{2},(\omega(n))^{-1}\right\},\qquad n+1\in N;\quad 1-z=n^{-1}\gamma(n).$$

It is easy to verify that (i) $1<a\leqslant 2$, and (ii) if n is sufficiently large ($n>n'$), then (43) holds. Setting $z=1-n^{-1}\gamma(n)$ in (66), we obtain that for $n>n'$ one has

$$1\leqslant A(p)((1-n^{-1}\gamma(n))^{-3/4}(\gamma(n))^{-1/2}+\omega(n)(\gamma(n))^{1/6}(1-n^{-1}\gamma(n))^{-5/12}).$$

Letting $n\to\infty$, we obtain a contradiction. In the cases $\frac{4}{3}<p<4$ and $p>4$ the assertions of the theorem follow directly from Lemma 2.

Proof of Theorem 3. Let $1\leqslant p<\frac{4}{3}$. We assume, contrary to the assertion of the theorem, that there exists $\varepsilon>0$ satisfying the stated condition. Setting $f=f_a$ in (60) and making use of (62), (64), and (65), under the conditions of Lemma 4 we obtain after some simplification

$$z^{3/2}(1-z)^{1/2}\leqslant A(p)((n+1)^{-1/2}z^{3/4}+(n+1)^{1/6+\varepsilon(2-2/p)}z^{13/12+\varepsilon}(1-z)^{2/3+2\varepsilon}). \tag{67}$$

We set

$$z=1-n^{(-p-12\varepsilon p+6\varepsilon)/p(1+12\varepsilon)}. \tag{68}$$

It is easy to verify that if n is sufficiently large ($n>n''$), then both conditions of Lemma 4 hold. By assuming $n>n''$, inserting the right-hand side of (68) into (67), and letting $n\to\infty$ we obtain a contradiction.

We consider the case $2 \leqslant p < 4$. The proof is again by contradiction, i.e., we assume that, contrary to the assertion of the theorem, there exists $\varepsilon > 0$ satisfying the stated condition. First of all we prove that

$$E_n(f_a)_2 \leqslant A(n+1)^{-1/2}(1-z)^{-1/2} z^n. \tag{69}$$

Taking into account (63), we obtain

$$E_n(f_a)_2 = 2\left(\sum_{k=n+1}^{\infty} \frac{z^{2k+1}}{2k+1}\right)^{1/2} = 2\left(\int_0^z \frac{t^{2n+2}}{1-t^2}\,dt\right)^{1/2}$$
$$\leqslant 2\left(\int_0^z \frac{t^{2n+2}}{1-t}\,dt\right)^{1/2} \leqslant A(n+1)^{-1/2}(1-z)^{-1/2} z^n,$$

which proves (69). From (69) and (42) for $2 \leqslant p < 4$ we obtain

$$E_n(f_a)_p \leqslant A(p)\, n^{1/2-2/p}(1-z)^{-1/2} z^n. \tag{70}$$

If in (61) we replace f by f_a and take into account (62), (64), and (70), then, after some simplification we obtain

$$1 \leqslant A(p)((n+1)^{-1/2}(1-z)^{-1/2} + (n+1)^{-1/2+\varepsilon+p/4-2\varepsilon/p}(1-z)^{-1/2+p/4+\varepsilon}). \tag{71}$$

We set

$$1-z = (n+1)^{(-p(-2+p+4\varepsilon)+4\varepsilon)/p(-2+p+4\varepsilon)}. \tag{72}$$

It is easy to see that if n is sufficiently large ($n > n'''$), then both conditions of Lemma 4 are satisfied. If we assume that $n > n'''$, if in (71) we replace $1-z$ with the right-hand side of (72), and if we let $n \to \infty$, then we arrive at a contradiction.

For the last case $\frac{4}{3} < p < 2$ we once more use proof by contradiction. Making use of (42) and (44), under the conditions of Lemma 4, we obtain

$$E_n(f_a)_p \leqslant A(p)\left\{(n+1)^{3/2-2/p}(a-1)^{1/4} z^m + (a-1)^{1/4} \sum_{v=n+1}^{\infty} v^{1/2-2/p} z^v\right\}$$
$$\leqslant A(p)\cdot\left((n+1)^{3/2-2/p}(a-1)^{1/4} z^m + (a-1)^{1/4}(n+1)^{1/2-2/p}\cdot\frac{z^{n+1}}{1-z}\right)$$
$$= A(p)(1-z)^{1/2} z^{n+3/4}(z(n+1)^{3/2-2/p} + (n+1)^{1/2-2/p}(1-z)^{-1}).$$

The rest of the proof is completely analogous to the previous proofs and, consequently, it will be omitted.

7. THE ULTRASPHERICAL CASE

There is a generalization of Theorem 1 to the ultraspherical case. We formulate it without proof.

For $-\frac{1}{2}<\alpha<\frac{1}{2}$ let $\{J_k^{(\alpha,\alpha)}\}_0^\infty$ be the system of ultraspherical polynomials, orthonormal on $[-1, 1]$ with respect to the weight function $(1-x^2)^\alpha$, and let $L_{p,\alpha}$ $(1\leqslant p<\infty)$ be the space of functions f, measurable on $[-1, 1]$ and integrable to the power p with respect to the weight $(1-x^2)^\alpha$, with norm

$$\|f\|_{p,\alpha}=\left\{\int_{-1}^{1} (1-x^2)^\alpha\,|f(x)|^p\,dx\right\}^{1/p}.$$

We set, by definition, $L_{\infty,\alpha}=C$ and $\|f\|_{\infty,\alpha}=\|f\|_C$. We introduce also the following notations:

$$c_k^{(\alpha)}(f)=\int_{-1}^{1} (1-x^2)^\alpha f(x)\,J_k^{(\alpha,\alpha)}(x)\,dx, \qquad k+1\in N;$$

$$S_n^{(\alpha)}(f)=\sum_{k=0}^{n} c_k^{(\alpha)}(f)\,J_k^{(\alpha,\alpha)}, \qquad n+1\in N;$$

$$R_n^{(\alpha)}(f)=f-S_n^{(\alpha)}(f), \qquad n+1\in N;$$

$$E_n(f)_{p,\alpha}=\inf\{\|f-Q_n\|_{p,\alpha}: Q_n\in H_n\}.$$

THEOREM 4. *Let $f\in C$, $n+1\in N$. Then*

$$\|R_n^{(\alpha)}(f)\|_C \leqslant \begin{cases} A(\alpha,p)(E_n(f)_C+(n+1)^{(2\alpha+2)(\alpha+1/2)/p(\alpha+3/2)}\,(E_n(f)_C)^{2/(2\alpha+3)} \\ \quad\times(E_n(f)_{p,\alpha})^{2(\alpha+1/2)/(2\alpha+3)}), \qquad 1\leqslant p<\dfrac{2\alpha+2}{\alpha+3/2}; \\ A(\alpha)\cdot(E_n(f)_C+(n+1)^{\alpha+1/2}\,(\ln(n+2))^{(\alpha+1/2)^2/(2\alpha+2)(\alpha+3/2)} \\ \quad\times(E_n(f)_C)^{1/(\alpha+3/2)}\,(E_n(f)_{p,\alpha})^{(\alpha+1/2)/(\alpha+3/2)}), \qquad p=\dfrac{2\alpha+2}{\alpha+3/2}; \\ A(\alpha,p)(E_n(f)_C+(n+1)^{\alpha+1/2}\,(E_n(f)_{p,\alpha})^{p(\alpha+1/2)/(2\alpha+2)} \\ \quad\times(E_n(f)_C)^{1-p(\alpha+1/2)/(2\alpha+2)}), \qquad \dfrac{2\alpha+2}{\alpha+3/2}<p<\dfrac{2\alpha+2}{\alpha+1/2}; \\ A(\alpha)(E_n(f)_C+(n+1)^{\alpha+1/2}\,(\ln(n+1))^{(\alpha+3/2)/(2\alpha+2)}\,E_n(f)_{p,\alpha}), \\ \quad p=\dfrac{2\alpha+2}{\alpha+1/2}; \\ A(\alpha,p)(E_n(f)_C+(n+1)^{\alpha+1/2}\,E_n(f)_{p,\alpha}), \qquad p>\dfrac{2\alpha+2}{\alpha+1/2}. \end{cases}$$

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